Mathematics: Analysis and Approaches SL

Chapter summaries

Haese Mathematics

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CHAPTER 1: THE BINOMIAL THEOREM

A Factorial notation
B Binomial expansions
C The binomial theorem

Syllabus reference: SL 1.9

In this chapter, students are introduced to factorial notation. This is important as it is encountered in several different contexts in the coming chapters. There is a Discussion regarding the definition of $0! = 1$. Students should find that this definition, whilst meaningless in the context of the original definition of factorial, is not "arbitrary", in that it was sensibly chosen in order that properties of factorial, such as $n! = n \times (n-1)!$, still apply. When considering other areas of mathematics where definitions are expanded in this way, they could discuss how the definition of exponents is expanded to include zero, negative, and rational exponents, or how the definition of trigonometric ratios is expanded to include angles larger than 90 degrees.

The counting principles of permutations and combinations are not included in the syllabus at SL. However we have included an Investigation on these topics, as we believe it is important for students to understand what ${}^nC_r$ represents. This allows students to gain an understanding of the binomial coefficients, and their link with Pascal's triangle.

This chapter will provide a good grounding for studying the binomial distribution in Chapter 20.

CHAPTER 2: QUADRATIC FUNCTIONS

A Quadratic functions
B Graphs of quadratic functions
C Using the discriminant
D Finding a quadratic from its graph
E The intersection of graphs
F Problem solving with quadratics
G Optimisation with quadratics
H Quadratic inequalities

Syllabus reference: SL 2.6, SL 2.7

This chapter follows from the work done in Chapter 4 of the Mathematics: Core Topics SL book involving quadratic equations.

We place this chapter ahead of the Functions chapter, so that when we get to the Functions chapter we can use quadratic functions as tools for studying domain, range, and composite and inverse functions.

In Section E, students solve quadratic inequalities graphically by identifying where one graph lies above or below another. An algebraic approach to quadratic inequalities is given in Section H. In this section, sign diagrams are introduced as tools to help solve quadratic inequalities. At this stage, they are taught without mention of points where the function is undefined, as this is not relevant for quadratics. The treatment of sign diagrams is completed in Chapter 3, where we draw sign diagrams for rational functions, and here we consider points at which the function is undefined.

CHAPTER 3: FUNCTIONS

A Relations and functions
B Function notation
C Domain and range
D Rational functions
E Composite functions
F Inverse functions
G Absolute value functions

Syllabus reference: SL 2.2, SL 2.3, SL 2.4, SL 2.5, SL 2.8

Although there is a significant amount of function material that is common to both courses, this Functions chapter has been put in the Mathematics: Analysis and Approaches SL book because these students must receive a more formal treatment of inverse functions. They must also study the reciprocal function and rational functions, which provides their first experience with asymptotes. This is where students learn how to indicate where a function is undefined on a sign diagram.

Students should be encouraged to construct sign diagrams for rational functions, as they are helpful for determining the behaviour of the function near its asymptotes.

Rather than having a section specifically devoted to graphing functions using technology, this work has been absorbed into the other sections of this chapter. For example, in Section C, students must use technology to find the turning points of a function, as well as the value of the function at the endpoints of the domain, to help determine the range of the function.

Section F (Inverse functions) does contain some questions involving functions like $f(x) = x^2$ for $x \geqslant 0$, however we do not consider this to be "domain restriction" (which is only required at HL) as the rule and the domain defines the function in its own right.

The final section covers the absolute value function, which will be useful for when students encounter calculus. The absolute value function is used to find the areas between curves with technology, and the absolute value sign occurs in the integral of $\ln x$.

CHAPTER 4: TRANSFORMATIONS OF FUNCTIONS

A Translations
B Stretches
C Reflections
D Miscellaneous transformations

Syllabus reference: SL 2.11

In this chapter we build on the function work done in Chapter 3 to consider transformations of functions. It is emphasised that the scale factors for the vertical and horizontal stretches must be positive: we should not consider the transformation from $y = x^2$ to $y = -2x^2$ as a vertical stretch with scale factor -2. Instead, it should be considered as a vertical stretch with scale factor 2, followed by a reflection in the x-axis. This allows us to preserve the uniqueness of tranformations.

In the Discussion in Section B, students should conclude that for $y = x^2$, the transformation by a *vertical* stretch with scale factor 4 is equivalent to a *horizontal* stretch with scale factor $\frac{1}{2}$. However, not every vertical stretch has an equivalent horizontal stretch in this manner. For example, consider a function which does not pass through the origin. A vertical stretch will change the y-intercept of the function, but a horizontal stretch cannot change the y-intercept.

We use the term "stretch" as given in the syllabus, but we also use the term "dilation", as it is a more general term which does not carry any particular geometric connotations. For example, it may be misleading for students to consider a vertical stretch with scale factor $\frac{1}{2}$, as this is a compression rather than a stretch.

Composite transformations are introduced as additional transformations are discovered, rather than as a section of its own.

In the Discussion at the end of Section C, students are asked for which combinations of transformations is the order in which the transformations are performed important. To get students started on this, it could be illustrated to them that if a function is to be reflected in the x-axis and then rotated, the order in which these occurs does matter, as you will obtain different results. As students dig deeper into this, it may be useful to consider the coordinates that are affected by the transformations. If one transformation only affects the x-coordinate, and one only affects the y-coordinate, then the transformations act *independently* of each other, and the order in which they are performed does not matter.

CHAPTER 5: EXPONENTIAL FUNCTIONS

- **A** Rational exponents
- **B** Algebraic expansion and factorisation
- **C** Exponential equations
- **D** Exponential functions
- **E** Growth and decay
- **F** The natural exponential

Syllabus reference: SL 2.9, SL 2.10

We begin the chapter with the study of rational exponents. This gives meaning to the value of a^x for all rational x, and lays the foundation for the study of exponential functions, which define a^x for all real x.

Students solve exponential equations by equating exponents. It should be pointed out to students that this can only be done for particular cases of exponential equations, where it is easy to write both sides of the equation in the same base. This will set the scene for being able to solve a wider range of exponential equations using logarithms in the next chapter.

Students will use the transformations studied in Chapter 4 to predict the shape of graphs such as $y = 2^x + 3$ and $y = 3^{x-1} - 4$.

In the Discussion at the end of Section D, students are asked why we specify a positive base number a. As a starting point, they should recognise that, since $a^{\frac{1}{2}} = \sqrt{a}$, a^x will be undefined at $x = \frac{1}{2}$ for negative values of a. By considering $y = (-2)^x$ for the integer values of x, students should realise that essentially two subgraphs start to form: the graph of $y = 2^x$ for even values of x, and $y = -(2^x)$ for odd values of x. When considering its domain, students should find that the function is undefined for all rational x with even denominator. Students may find the graphing package useful for exploring functions of this type.

CHAPTER 6: LOGARITHMS

- **A** Logarithms in base 10
- **B** Logarithms in base a
- **C** Laws of logarithms
- **D** Natural logarithms
- **E** Logarithmic equations
- **F** The change of base rule
- **G** Solving exponential equations using logarithms
- **H** Logarithmic functions

Syllabus reference: SL 1.7, SL 2.9

At the very start of the chapter, we introduce logarithmic functions as the inverse of exponential functions, to motivate their use. We then proceed through the standard properties of logarithms, before returning to logarithmic functions at the end of the chapter.

In Section G, students learn how to use logarithms to solve any exponential equation. Where necessary, we use the logarithm laws to write the solution in terms of base 10 logarithms, which can then be approximated using a calculator. The growth and decay questions first posed in Chapter 5 are now extended to answer questions like “how long will it take for the population to reach 500?”.

We would expect that classes will have completed this chapter by the end of the first year.

CHAPTER 7: THE UNIT CIRCLE AND RADIAN MEASURE

- **A** Radian measure
- **B** Arc length and sector area
- **C** The unit circle
- **D** Multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$
- **E** The Pythagorean identity
- **F** Finding angles
- **G** The equation of a straight line

Syllabus reference: SL 3.4, SL 3.5, SL 3.6

We start this chapter with an introduction to radian measure. The arc length and sector area have already been covered in the Mathematics: Core Topics SL book, but with the angle given in degrees. We now present the much simpler formulae when the angle is given in radians.

In this chapter we consider the complete unit circle, allowing the students to give meaning to the trigonometric ratios for *any* angle. Students should be encouraged to become familiar with the identities involving supplementary, complementary, and negative angles, as an understanding of these will help with the formulation of trigonometric functions in the following chapter.

Section C contains a Discussion about identities. Students should be able to distinguish between equations, which are true for only particular values of a variable, and identities, which are true for all *all* values of the variable. Examples of identities include the Pythagorean identity, as well as expansion and factorisation identities such as $(a+b)(a-b) = a^2 - b^2$.

In Section F, we use the inverse trigonometric ratios, as well as the properties of supplementary and negative angles, to find angles with a given trigonometric ratio, over a particular interval. Mastering this work requires a good understanding of the angle the calculator gives when calculating an inverse trigonometric ratio, as well as the ability to find other angles with the same trigonometric ratio. This work will provide a solid grounding for solving trigonometric equations in Chapter 9.

CHAPTER 8: TRIGONOMETRIC FUNCTIONS

- **A** Periodic behaviour
- **B** The sine and cosine functions
- **C** General sine and cosine functions
- **D** Modelling periodic behaviour
- **E** The tangent function

Syllabus reference: SL 3.7

Here we use the transformations studied in Chapter 4 to construct graphs of trigonometric functions. We use the general form $y = a \times \sin(b(x-c)) + d$ rather than $y = a \times \sin(b(x+c)) + d$ as given in the syllabus, so that we can talk sensibly about a horizontal translation of c units and a vertical translation of d units. This is consistent with the transformation $f(x-a)$ described in the Functions section of the syllabus.

In Section D, we use given information to find a trigonometric model for a situation. We may have particular information about the situation (such as the maximum and minimum values, and the time between these values), or we may have data points which display periodic behaviour. In the latter case, we can use technology to check our regression models, but students should be aware that the calculator may not give the model in the form $y = a \times \sin(b(x-c)) + d$, and the value of c may be different from the one students obtained. However, students should be able to use the trigonometric identities to confirm that their answer and the answer given by their calculator are equivalent.

CHAPTER 9: TRIGONOMETRIC EQUATIONS AND IDENTITIES

- **A** Trigonometric equations
- **B** Using trigonometric models
- **C** Trigonometric identities
- **D** Double angle identities

Syllabus reference: SL 3.6, SL 3.8

This chapter builds on the work done in Chapter 7 and 8. Students solve trigonometric equations both graphically and analytically. When solving trigonometric equations, students should be encouraged to pay careful attention to the interval over which solutions must be found. Trigonometric equations are also applied to the trigonometric models studied in Chapter 8.

Students must also use trigonometric identities to simplify expressions and solve equations. It would be beneficial for students to understand how identities are derived, rather than trying to memorise them. For example, they should understand that $\cos x = \cos(-x)$ can be established by a reflection of points on the unit circle in the x-axis, and that the Pythagorean identity can be used to find the equivalent expressions for $\cos 2x$.

The double angle identities for cosine and sine are presented. The double angle identity for $\tan$ is not required in the syllabus, but it is straightforward to derive from the identities for cosine and sine. Students are asked to do this as an exercise question.

CHAPTER 10: REASONING AND PROOF

- **A** Logical connectives
- **B** Proof by deduction
- **C** Proof by equivalence
- **D** Definitions

Syllabus reference: SL 1.6

This chapter is placed here to give students a break between previous functions chapters, and the calculus chapters to come. Logical connectives are discussed as they are necessary to understand how to construct a sound argument, with one statement leading to the next. The concept of definitions is important as you need them as basis of a proof, and without them, you cannot really prove anything.

We have steered clear of "geometric style" proofs, as these are not mentioned in the syllabus, and students are likely to have encountered them in previous years.

We have distinguished between "proof by implication", in which only the correct implication from one step to the next is required, and "proof by equivalence", in which each deductive step must be an equivalence, allowing us to establish the equivalence of two statements.

Counter examples are mentioned briefly, but will be covered in more detail at HL. A Discussion is included to highlight the distinction between labelling proofs as "direct" and "indirect", and labels like "proof by equivalence", and so on.

CHAPTER 11: INTRODUCTION TO DIFFERENTIAL CALCULUS

- **A** Rates of change
- **B** Instantaneous rates of change
- **C** Limits
- **D** The gradient of a tangent
- **E** The derivative function
- **F** Differentiation from first principles

Syllabus reference: SL 5.1

This chapter provides students with their first look at differential calculus. Although much of the calculus content is common between the SL courses, we expect the classes will be separated by the time they encounter calculus. Having the calculus chapters in the separate books allows a more targeted approach to calculus for each course.

This chapter begins with rates of change, which is used to motivate an informal study of limits. We study limits at infinity, as this will be used in the motivation of the natural exponent e in the following chapter. It will also be used in the study of integration.

Although differentiation from first principles is not explicitly in the syllabus, we feel that it is essential for understanding the process of differentiating a function, and there is no point in studying limits if we do not consider first principles.

At the end of the chapter, there is a Discussion which asks whether a function always has a derivative function, and whether the domain of a function and its derivative are always the same. To help guide students, they should be encouraged to consider the definition of "function" as broadly as possible, as a collection of disconnected points is considered to be a function, as long as no two share the same x-coordinate. The function need not have an "equation" which defines its set of points. Students should also consider functions such as $f(x) = \sqrt{x}$, and discuss whether this function has a limit as x approaches 0, given that the function is undefined for $x < 0$.

CHAPTER 12: RULES OF DIFFERENTIATION

- **A** Simple rules of differentiation
- **B** The chain rule
- **C** The product rule
- **D** The quotient rule
- **E** Derivatives of exponential functions
- **F** Derivatives of logarithmic functions
- **G** Derivatives of trigonometric functions
- **H** Second derivatives

Syllabus reference: SL 5.3, SL 5.6, SL 5.7

In this chapter students will discover rules for differentiating functions. We use limits at infinity to motivate the natural exponential function e^x as the function which is its own derivative. Many of the proofs of these rules, while not required, are included to aid the student's understanding.

The derivative of $\tan x$ is not required in the syllabus, but can be easily derived from the derivatives of $\sin x$ and $\cos x$ using the quotient rule. Students are asked to do this as an exercise.

Throughout this chapter, students should be encouraged to remember what a derivative function means, rather than just performing the differentiation without giving thought to its meaning. To this end, we have included exercises asking students to answer questions involving the gradients of tangents to the functions they are differentiating.

CHAPTER 13: PROPERTIES OF CURVES

- **A** Tangents
- **B** Normals
- **C** Increasing and decreasing
- **D** Stationary points
- **E** Shape
- **F** Inflection points
- **G** Understanding functions and their derivatives

Syllabus reference: SL 5.2, SL 5.4, SL 5.7, SL 5.8

This chapter allows students to apply the calculus they have learnt to discover the properties of curves.

In the previous Mathematics: Analysis and Approaches SL book, we dealt with tangents and normals together in a single section, however we have placed them in separate sections here as the tangents section is quite large, and there are sufficient concepts presented in this topic to warrant a section of its own.

Students should be encouraged to think of the concepts of increasing and decreasing in terms of intervals, rather than at a particular point. This will help students understand why, for example, the graph of $y = x^2$ is increasing for $x \geqslant 0$, and decreasing for $x \leqslant 0$.

In Section D we return to sign diagrams, and see how the sign diagram of $f'(x)$ can be used to determine the nature of the stationary points of $f(x)$.

We have added a section dealing with shape before we introduce inflection points. This is similar to how we talk about increasing and decreasing functions before we deal with stationary points.

CHAPTER 14: APPLICATIONS OF DIFFERENTIATION

A Rates of change
B Optimisation

Syllabus reference: SL 5.1, SL 5.8

In this final chapter of differential calculus, we explore some of its real world applications. The important skill in this chapter is to take the calculus techniques learnt in previous chapters, and to apply them to real life problems, and to interpret the results in the context of that problem.

Students should keep in mind the constraints imposed by the context of the problem, and to make sure their solution makes sense in this context. This is especially true in problems involving trigonometry, in which we must remember that, for example, $\sin x$ can only take values from -1 to 1.

The chapter ends with an Activity about cubic spline interpolation. This could potentially be a useful starting point for a student's Mathematical Exploration.

CHAPTER 15: INTRODUCTION TO INTEGRATION

A Approximating the area under a curve
B The Riemann integral
C Antidifferentiation
D The Fundamental Theorem of Calculus

Syllabus reference: SL 5.5, SL 5.11

We begin our study of integration by calculating the area under the curve, using the idea of limits. We feel this approach is consistent with how integral calculus was developed historically. Limits at infinity are again used here to explore what happens as we consider more and more upper and lower rectangles.

The title in Section B may appear intimidating to students, but all that is happening here is that we are using the integral symbol as notation to denote the area under a curve. It will be beneficial for students to understand this notation before moving on to see how the area under a curve relates to the antiderivative of a function.

We then move on to consider antiderivatives of funcitons, culminating in the Fundamental Theorem of Calculus, which links antiderivatives and the area under a curve.

This is a short chapter, but is quite involved conceptually, so it is important that students spend the time to understand the link between antiderivatives and the area under a curve.

CHAPTER 16: TECHNIQUES FOR INTEGRATION

A Discovering integrals
B Rules for integration
C Particular values
D Integrating $f(ax+b)$
E Integration by substitution

Syllabus reference: SL 5.5, SL 5.10

Now that we have established that integration is the reverse process of differentiation, we use the rules for differentiation in reverse to develop the rules for integration. At this stage we only consider indefinite integration.

At the end of Section B, students are asked to discuss why we specify that our rule for integrating x^n is not valid for $n = -1$. Students should conclude that substituting $n = -1$ into this rule would result in a division by zero. They should also note that the case $n = -1$ is considered separately, and the integral of $\frac{1}{x}$ is $\ln|x| + c$.

In some instances, extra information about the original function is included, allowing us to determine the constant of integration. Occasionally simultaneous equations will be required to find multiple unknowns.

CHAPTER 17: DEFINITE INTEGRALS

- **A** Definite integrals
- **B** The area under a curve
- **C** The area above a curve
- **D** The area between two functions
- **E** Problem solving by integration

Syllabus reference: SL 5.5, SL 5.11

Now that we have established the rules for integration, we have the tools to calculate definite integrals, and to explore the relationship between definite integrals and areas.

Areas under and above curves are treated separately, giving students more of an opportunity to see that the definite integral is a *signed* area function. This approach also allows students to practise with areas above curves in a section in its own right, rather than as a special case of area between two curves $f(x)$ and $g(x)$ where $f(x) = 0$.

The syllabus specifies that some definite integrals will not be able to be performed analytically, and so technology must be used. To this end, we have included calculator instructions, screenshots, and exercises which allow students to practise finding definite integrals using technology.

When the integration requires substitution, we do not ask students at SL to transform the endpoint values in the definite integral. Instead, indefinite integration will be performed, and then the definite integral will be applied.

The chapter ends with an Activity about Buffon's needle problem. Some of the integration involved, particularly obtaining the probability given in **3 b** of the long needle case, is quite difficult, so it should not cause students concern if they cannot do this. However, it is hoped that the more able student will find this Activity interesting and engaging.

CHAPTER 18: KINEMATICS

- **A** Displacement
- **B** Velocity
- **C** Acceleration
- **D** Speed

Syllabus reference: SL 5.9

Having dealt with both differentiation and integration, we will now bring these processes together in the study of kinematics. Rather than dealing first with differentiation, then with integration, we will deal with particular concepts of kinematics, and present the differentiation and integration aspects of each concept together.

Most students are likely to have previously encountered questions concerning motion in the form of travel graphs. A difficulty students are likely to encounter in this chapter is talking about displacement rather than distance, and velocity rather than speed. For this reason, we have included a brief outline of the language of motion at the start of the chapter.

In Example 4 part **b**, we are given a velocity function, and asked to find the distance travelled in the first 4 seconds. In answering this question, we choose the initial displacement to be zero. We do this because an initial displacement is not given in the question, and the choice of initial displacement of the particle does not affect its distance travelled. Setting an initial displacement allows us to perform the distance calculations without involving a constant of integration c.

In Section D, a Discussion asks students to explain why the "sign test" for speed works, by considering various scenarios. The students should use the fact that the speed is the magnitude of the velocity. So, for example, if the velocity and acceleration are both negative, this means that the velocity is negative, and its value is decreasing, which means its *magnitude* is increasing, and so its speed is increasing.

CHAPTER 19: BIVARIATE STATISTICS

- **A** Association between numerical variables
- **B** Pearson's product-moment correlation coefficient
- **C** Line of best fit by eye
- **D** The least squares regression line
- **E** The regresion line of x against y

Syllabus reference: SL 4.4, SL 4.10

Most of this chapter is similar in style and presentation as in previous books and the MYP series.

In Section E (The least squares regression line), we delve deeper into the regression procedure and provide formulae for the regression coefficients. An online-only derivation of the coefficients is provided for the students who are more algebraically inclined. Of course in exercise questions, students are encouraged to use technology to find the coefficients.

Section F (The regression line of x against y) is new. We motivate the need for the line of x against y in terms of accuracy/precision and make reference to the variable with which the error of the measurement is associated (and hence the *direction* that the distance from the regression line is measured).

CHAPTER 20: DISCRETE RANDOM VARIABLES

- **A** Random variables
- **B** Discrete probability distributions
- **C** Expectation
- **D** The binomial distribution
- **E** Using technology to find binomial probabilities
- **F** The mean and standard deviation of a binomial distribution

Syllabus reference: SL 4.7, SL 4.8

We start the chapter with an introduction to the concept of (discrete) random variables and their probability distributions. If you are going through the Mathematics: Core Topics SL book and this book in chapter order, it will have been a long time since the students have seen probability. Before starting this chapter, it may be beneficial to briefly revise key probability concepts as they are assumed throughout this chapter and Chapter 21 (The normal distribution).

Section C (Expectation) continues on directly from Section J (Making predictions with probability) from Chapter 10 of the Mathematics: Core Topics SL book.

In Section C.2, a Discussion asks students whether we would expect a gambling game to be "fair". Students should recognise that we would not expect gambling games to be "fair", otherwise the operator of the game would not make a profit. A useful direction to lead students would be to ask whether the word "fair" in the mathematical sense is equivalent to how the word is used in everyday life, and whether the fact that gambling games are not "fair" implies that the operators are being underhanded or deceptive.

Combinations and binomial coefficients are essential for the introduction of the binomial distribution and the formulation of its probability mass function. So ensuring that students are familiar with binomial coefficient notation is important.

CHAPTER 21: THE NORMAL DISTRIBUTION

- **A** Introduction to the normal distribution
- **B** Calculating probabilities
- **C** The standard normal distribution
- **D** Quantiles

Syllabus reference: SL 4.9, SL 4.12

In the chapter's introduction, we briefly mention probability density functions as the continuous analogue of probability mass functions. Here, we give the definition that the probability is the area under the curve (and hence definite integral) without further comment or investigation. We are simply using the probability density function as a tool to justify the notion that area under the normal curve = probability later in the chapter, and nothing else.

The first section introduces the normal distribution by focusing on how it arises and exploring its *shape*. Probability calculations are treated separately in the following section. The standard normal distribution and quantiles sections have not seen much change from the previous Mathematics SL book.